Last time: consider a matrix equation

$$Ax = b$$

with associated homogeneous matrix equation

$$Ax = 0$$

Then $\exists$ one-to one correspondence

$$\{\text{solutions of } Ax=b\} \underset{\text{add } s_b}{\overset{\text{subtract } s_b}{\rightleftarrows}} \{\text{solutions of } Ax=0\}$$

for any particular solution s_b of $Ax=b$

In other words: if s is solution of $Ax=b$

then $s_h = s - s_b$ is a solution of $Ax=0$

and if s_h is a solution of $Ax=0$

then $s = s_b + s_h$ is a solution of $Ax=b$

Linear independence (of vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$)

$\{v_1, \dots, v_m\}$ are called

linearly dependent if $\exists$ $c_1, \dots, c_m \in \mathbb{R}$ not all 0 s.t.

• linearly dependent if $\exists\, c_1, \dots, c_m$

$$c_1 v_1 + \dots + c_m v_m = 0$$

• linearly independent if $\forall\, c_1, \dots, c_m \in \mathbb{R}$ not all 0

$$c_1 v_1 + \dots + c_m v_m \neq 0$$

(geometrically, linear independance means that span$\{v_1, \dots, v_m\}$ has dimension exactly $= m$)

How to test linear independance?

Let $A = \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}$

trivial solution

$X = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$

$c_1 v_1 + \dots + c_m v_m$

$\parallel$

If $Ax = 0$ has a solution other than $x = 0$, then $v_1, \dots, v_m$ are linearly dependent; otherwise, they are linearly independent

THM 5.1 : let $v_1, \dots, v_m$ be vectors in $\mathbb{R}^n$

they are linearly dependent

one of the v_j's is a linear combination of the others

<u>Prop</u>: • if $m > n$, then any set of vectors $\{v_1, \dots, v_m\}$ are linearly dependent

• any set of vectors among whom is 0 (i.e. $\{v_1, \dots, v_{k-1}, 0, v_{k+1}, \dots, v_m\}$) are linearly dependent

<u>Prop</u>: Given vectors $v_1, \dots v_m, w \in \mathbb{R}^n$

• if $\{v_1, \dots v_m\}$ are linearly dependent then $\{v_1, \dots, v_m, w\}$ are linearly dependent

• if $\{v_1, \dots, v_m, w\}$ are linearly independent, then $\{v_1, \dots, v_m\}$ are linearly independent

Proof of Thm 5.1

$\{v_1, \dots, v_m\}$ dependent means $\exists\ c_1, \dots, c_m$ not all 0 s.t.

$$c_1 v_1 + \dots + c_m v_m = 0$$

$\Longleftrightarrow \exists \; c_j \neq 0$ and so $\frac{c_1}{c_j} v_1 + \dots + \frac{c_{j-1}}{c_j} v_{j-1} + v_j + \frac{c_{j+1}}{c_j} v_{j+1} + \dots + \frac{c_m}{c_j} v_m = 0$

$\Longleftrightarrow \exists$ som j s.t. $v_j = -\frac{c_1}{c_j} v_1 - \dots - \frac{c_{j-1}}{c_j} v_{j-1} - \frac{c_{j+1}}{c_j} v_{j+1} - \dots - \frac{c_m}{c_j} v_m$

choose $a_k = -\frac{c_k}{c_j}$

$\exists \; a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m$ s.t.
$v_j = a_1 v_1 + \dots + a_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \dots + a_m v_m$ means some v_j is a linear combination of the others $(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$

called 1.10 in ex sheet

THM 5.2: $Ax = b$ has (assume A is $m \times n$)

(1) at least one solution $x \in \mathbb{R}^n$, $\forall \, b \in \mathbb{R}^m$ iff
span{columns of A} $= \mathbb{R}^m$

(2) at most one solution $x \in \mathbb{R}^n$, $\forall \, b \in \mathbb{R}^m$ iff
the columns of A are linearly independent

Proof of (1): last week

Proof of (2): if $\exists$ a solution s_b of $Ax = b$,

$$|\{\text{solutions of } Ax = b\}| = |\{\text{solutions of } Ax = 0\}| \quad (*)$$

Recall that $Ax = 0$ always has one solution $(x = 0)$ and it has more solutions $\Longleftrightarrow$ columns of A are dependent

So columns are independent $\Longleftrightarrow |\{\text{solutions of } Ax = 0\}| = 1$

$\overset{(*)}{\Longleftrightarrow} |\{\text{solutions of } Ax = b\}| \leq 1$

($\forall$ $\quad$ + $\quad$ S $\quad$ $|S|$ = number of elements)

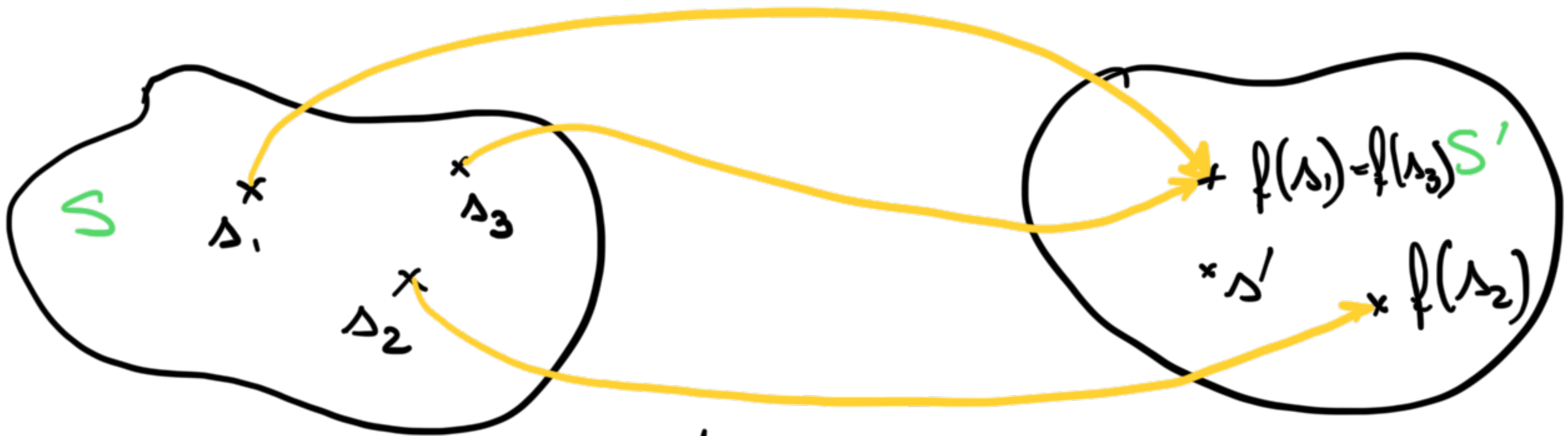
$\left(\forall \text{ set } S,\ |S| = \quad \text{in } S \right)$

"size", "cardinality"

DEF 5.3: given sets S and S', a function

$$f : \underset{\text{domain}}{S} \longrightarrow \underset{\text{codomain / target}}{S'}$$

is an assignment of some $f(s) \in S'$, $\forall s \in S$



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$$\mathrm{Im}(f) = \{ s' \in S' \mid \exists s \in S \text{ s.t. } f(s) = s' \} \subseteq S'$$

Examples:

(wacky): $\{\text{goat, triceratops, unicorn, narwal}\} \xrightarrow{f} \mathbb{N}$ (natural numbers)

$f(\text{animal}) = \#$ of horns of animal, $\mathrm{Im}\, f = \{1, 2, 3\}$

(arithmetic): $f : \mathbb{Z} \longrightarrow \{0, 1, 2, 3, 4, 5, 6\}$ ($\mathbb{Z}$: integers)

$f(k) = (\text{remainder of } k \text{ upon division by } 7)$, $\forall k \in \mathbb{Z}$

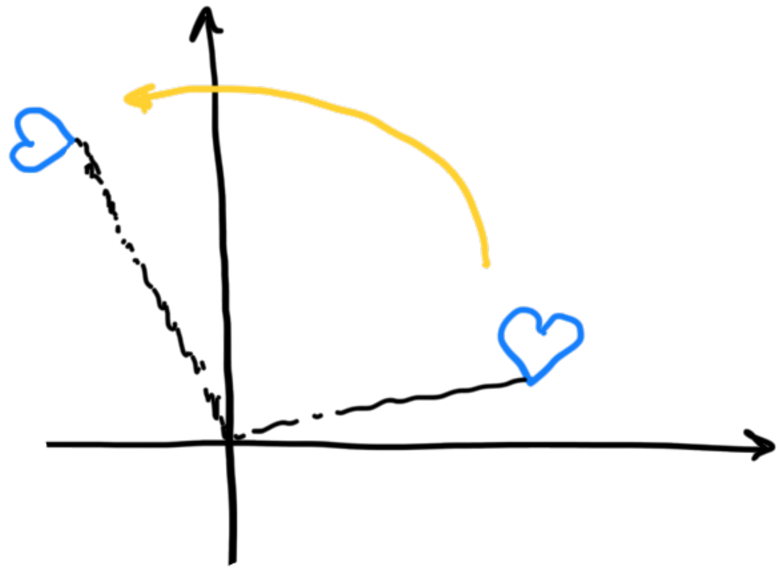
(analytic): $f : \mathbb{R} \longrightarrow \mathbb{R}$ ($\mathbb{R}$: real numbers) (analytic) : $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = e^{\sin\left(\frac{x}{\pi} + \sqrt{x^{19}} \ldots\right)}, \quad \forall x \in \mathbb{R}$$

(linear) : $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 7x$, $\forall x \in \mathbb{R}$

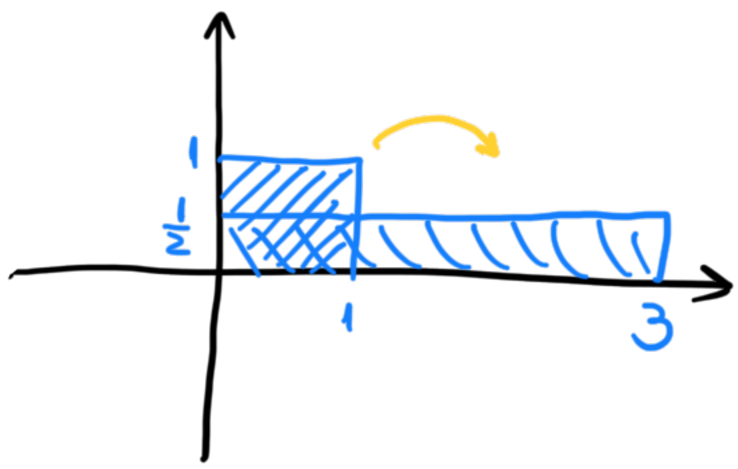
(geometric) :

ROTATION (by $90°$)



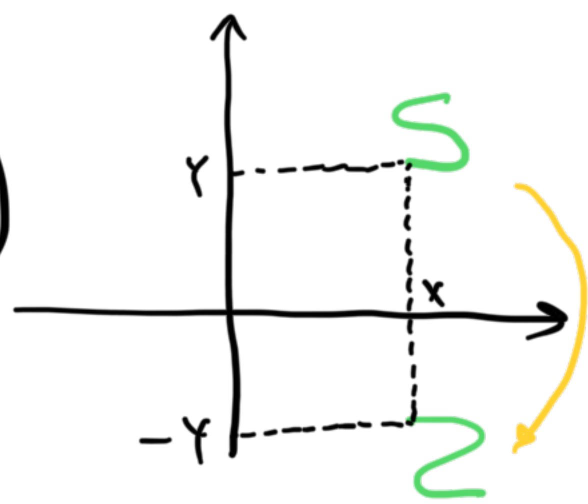
$$f: \mathbb{R}^2 \to \mathbb{R}^2$$
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

RESCALING (3 times horizontally, $\frac{1}{2}$ times vertically)



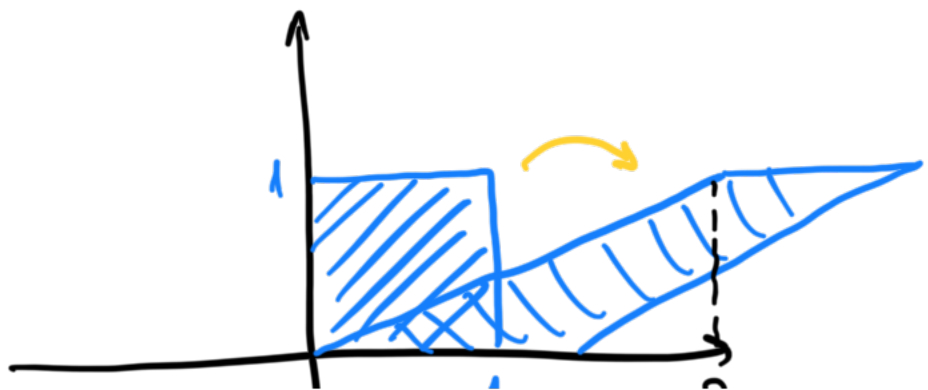
$$f: \mathbb{R}^2 \to \mathbb{R}^2$$
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ \frac{1}{2}y \end{pmatrix}$$

REFLECTION (across horizontal)

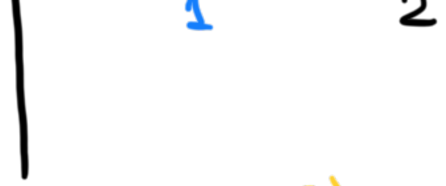


$$f: \mathbb{R}^2 \to \mathbb{R}^2$$
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

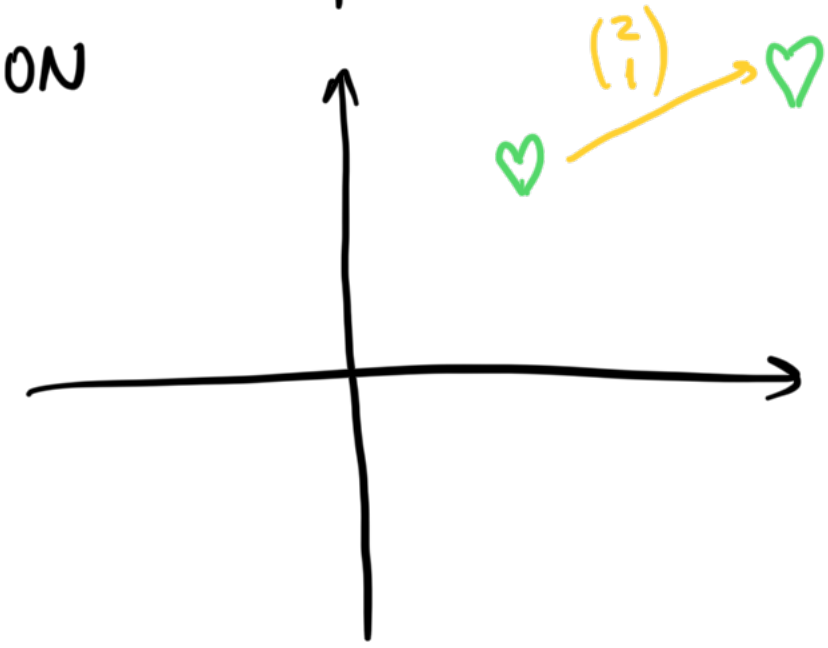
SHEARING



$$f: \mathbb{R}^2 \to \mathbb{R}^2$$
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ y \end{pmatrix}$$



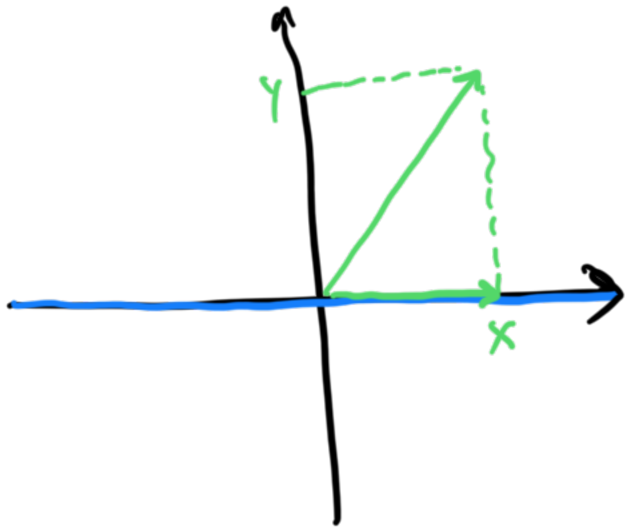
TRANSLATION (SHIFT)



$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2 \\ y+1 \end{pmatrix}$

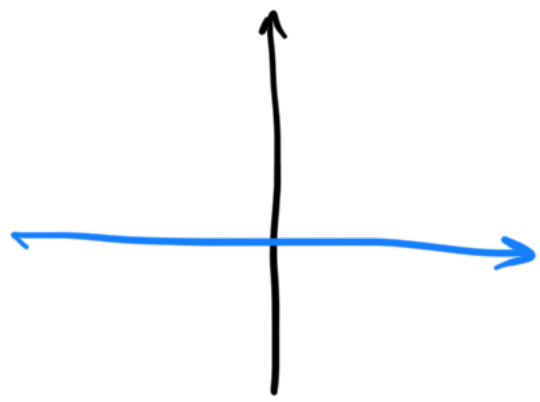
PROJECTION



$f: \mathbb{R}^2 \to \mathbb{R}^1$

$f\begin{pmatrix} x \\ y \end{pmatrix} = (x)$

INCLUSION



$f: \mathbb{R}^1 \to \mathbb{R}^2$

$f(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$

All functions above preserve parallelograms

All except translation preserve the origin

Preserving parallelograms means the following

f sends v_1+v_2 to $f(v_1)+f(v_2)$

DEF 5.4: a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called linear if it preserves vector addition & vector scaling, i.e.

① $f(v_1+v_2) = f(v_1)+f(v_2)$

② $f(cv) = c f(v)$, $\forall\, v_1, v_2, v \in \mathbb{R}^n$, $\forall\, c \in \mathbb{R}$

Ex (rotation is linear)

$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \xrightarrow{f} f(v_1) = \begin{pmatrix} -y_1 \\ x_1 \end{pmatrix}$

$v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \longrightarrow f(v_2) = \begin{pmatrix} -y_2 \\ x_2 \end{pmatrix}$

$v_1+v_2 = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \xrightarrow{f} f(v_1)+f(v_2) = \begin{pmatrix} -y_1-y_2 \\ x_1+x_2 \end{pmatrix}$

Properties of linear functions:

- $f(c_1v_1 + c_2v_2) = c_1 f(v_1) + c_2 f(v_2)$ ← this is equiv to axioms ① and ② combined

$\|$ ①

$f(c_1v_1) + f(c_2v_2) \overset{②}{=} c_1 f(v_1) + c_2 f(v_2)$

- $f(c_1v_1+\dots+c_nv_n) = c_1 f(v_1)+\dots+c_n f(v_n)$

$$f(c_1v_1+\dots+c_nv_n) \overset{①}{=} f(c_1v_1+\dots+c_{n-1}v_{n-1})+f(c_nv_n) \overset{①}{=} f(c_1v_1+\dots+c_{n-2}v_{n-2})+f(c_{n-1}v_{n-1})+f(c_nv_n) \overset{①}{=} \dots \overset{①}{=} f(c_1v_1)+\dots+f(c_nv_n) \xlongequal{\text{② n times}} c_1f(v_1)+\dots+c_nf(v_n)$$

- $f(0)=0$ (translations are not linear)

Proof using ②: take any $v\in\mathbb{R}^n$;

$$\underbrace{f(0)=f(0\cdot v)}_{\text{basic vector stuff}} \overset{②}{=} \underbrace{0\cdot f(v)=0}_{\text{basic vector stuff}}$$

$$\underset{\text{number}}{0}\,v=\underset{\text{vector}}{0}$$

Proof using ①: $f(0+0)\overset{①}{=}\cancel{f(0)}+f(0)$

$\quad\quad\quad\quad\quad\quad\cancel{f(0)}$ $\Rightarrow 0=f(0)$

Geometrically, linear functions send subspaces to subspaces

THM 5.5: any linear function

$$f:\mathbb{R}^n\longrightarrow\mathbb{R}^m$$

can be written uniquely as

$$f(x) = Ax \quad , \forall x \in \mathbb{R}^n$$

for some $m \times n$ matrix A depending on f

Conversely: all functions of this sort are linear

Examples:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ \frac{x_2}{2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies A = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1) = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies A = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$f(x_1) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(x_1) \implies A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Proof of converse: $\forall$ $m \times n$ matrix A, the function

$$f: \mathbb{R}^n \to \mathbb{R}^m, \quad f(x) = Ax \quad , \forall x \in \mathbb{R}^n$$

is linear, i.e. $f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2)$ axiom ① and ② combined

$$\begin{array}{ccc} \| & & \| \\ A(c_1 v_1 + c_2 v_2) & \underset{\text{last week}}{=\!=\!=} & c_1 A v_1 + c_2 A v_2 \end{array}$$

Other half of theorem: $\forall$ linear function $f: \mathbb{R} \to \mathbb{R}$,

we will construct an $m \times n$ matrix A s.t. such that

$$f(x) = Ax, \ \forall x \in \mathbb{R}^n$$

TBD next time